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Two-Dimensional Windows

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Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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TWO-DIMENSIONAL WINDOWS

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ABSTRACT

Two-dimensional windows find applications in many diverse fields,
such as the spectral estimation of random fields, the design of two-
dimensional digital filters, optical apodization, and antenna array
design. Many good one-dimensional windows have been devised, but
relatively few two-dimensional windows have been investigated. In this
paper we show that good two-dimensional windows can be obtained by
rotating good one-dimensional windows. That is, if w(x) is a good
symmetrical one-dimensional window, then $w_2(x, y) = w(\sqrt{x^2 + y^2})$
is a good circularly symmetrical two-dimensional window.

TWO-DIMENSIONAL WINDOWS

I. Introduction

Two-dimensional windows find applications in many diverse fields, such as the spectral estimation of random fields, the design of two-dimensional digital filters, optical apodization, and antenna array design.

Many good one-dimensional windows have been devised,[1-4] however, relatively few two-dimensional windows have been investigated.[5,6] In this paper, we establish a result which enables us to get good two-dimensional windows from good one-dimensional windows.

II. The Problem

We first review briefly the one-dimensional problem. Let the Fourier transform of a function $f(x)$ be $F(u)$. For some reason, we want to truncate $f(x)$:

$$g(x) = f(x) \, w(x) \tag{1}$$

$$w(x) = 0, \text{ for } |x| > A \tag{2}$$

where A is a constant.

The Fourier transform of $g(x)$ is

$$G(u) = F(u) \otimes W(u) \tag{3}$$

where $W(u)$ is the Fourier transform of $w(x)$, and $\otimes$ denotes convolution. Our problem is to choose an appropriate shape for the window function $w(x)$ such that $G(u)$ is close to $F(u)$ and in any region surrounding a discontinuity of $F(u)$, $G(u)$ will not contain excessive ripples. It is well known that the Fourier transform $W(u)$ of a good window $w(x)$ should have a big central peak and small sidelobes.

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In two dimensions, the problem is entirely similar. Let the two-dimensional Fourier transform of a function $f_2(x, y)$ be $F_2(u, v)$, and let

$$g_2(x, y) = f_2(x, y) \, w_2(x, y) \tag{4}$$

where
$$w_2(x, y) = 0 \text{ for } |x^2 + y^2| > A^2 \tag{5}$$

The Fourier transform of $g_2(x, y)$ is

$$G_2(u, v) = F_2(u, v) \otimes W_2(u, v) \tag{6}$$

where $W_2(u, v)$ is the Fourier transform of $w_2(x, y)$. The problem is to choose an appropriate shape for the two-dimensional window function $w_2(x, y)$ such that $G_2(u, v)$ is close to $F_2(u, v)$ and in the neighborhood of a discontinuity of $F_2(u, v)$, $G_2(u, v)$ does not contain excessive ripples.

III.   The Result

Intuitively, we feel that if $w(x)$ is a good symmetrical one-dimensional window, then
$$w_2(x, y) = w(\sqrt{x^2 + y^2}) \tag{7}$$

will be a good two-dimensional window.

This is indeed partially verified by the following two examples.

The first example is

$$w(x) = \begin{cases} 1, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases} \tag{8}$$

Then the Fourier transform is

$$W(u) = \frac{2 \sin u}{u} \tag{9}$$

whose first side-lobe peak is about 20% of the peak at $u = 0$. The corresponding two-dimensional window

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$$w_2(x,y) = \begin{cases} 1, & \text{for } |x^2 + y^2| \le 1 \\ 0, & \text{for } |x^2 + y^2| > 1 \end{cases} \tag{10}$$

has the two-dimensional Fourier transform

$$W_2(u,v) = \frac{2\pi J_1 (\sqrt{u^2 + v^2})}{\sqrt{u^2 + v^2}} \tag{11}$$

whose first side-lobe peak is only about 12¢ of the peak at $u = 0 = v$.

The second example is

$$w(x) = \begin{cases} 1 - x, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases} \tag{12}$$

The Fourier transform is

$$W(u) = \left(\frac{\text{Sin } \frac{u}{2}}{\frac{u}{2}} \right)^2 \tag{13}$$

whose side-lobe peak is about 4¢ of the peak at $u = 0$. The corresponding two-dimensional window

$$w_2(x,y) = \begin{cases} 1 - \sqrt{x^2 + y^2}, & \text{for } |x^2 + y^2| \le 1 \\ 0, & \text{for } |x^2 + y^2| > 1 \end{cases} \tag{14}$$

has a two-dimensional Fourier transform

$$W_2(u,v) = 2\pi \left[\rho^{-3} \int_0^\rho J_0(t)dt - \rho^{-2} J_0(\rho) \right] \tag{15}$$

where $\rho = \sqrt{u^2 + v^2}$, whose first side-lobe peak is only about 2¢ of the peak at $u = 0 = v$.

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The above comparisons are, however, unfair. Because when we convolve a window with a discontinuity, what count in the one-dimensional case are the areas under the side-lobes, while in the two-dimensional case what count are the volumes under the side-lobes. A fair comparison would be to look at the result of the convolution. One thing we can say along this line is contained in the following theorem which is the main result of this paper.

Theorem. If a symmetrical one-dimensional window $w(x)$ and a two-dimensional window $w_2(x, y)$ are related by

$$w_2(x, y) = w \ (\sqrt{x^2 + y^2}) \tag{16}$$

then their Fourier transforms $W(u)$ and $W_2(u, v)$ satisfy the relation

$$\frac{1}{2\pi} \ W_2(u, v) \otimes H_2(u, v) = W(u) \otimes H(u) \tag{17}$$

where

$$H(u) = \begin{cases} 1, & \text{for } u \geq 0 \\ 0, & \text{for } u < 0 \end{cases} \tag{18}$$

$$H_2(u, v) = \begin{cases} 1, & \text{for } u \geq 0 \text{ and all } v \\ 0, & \text{for } u < 0 \text{ and all } v \end{cases} \tag{19}$$

and $\otimes$ denotes convolution.

Proof. We first show that

$$W(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv \ W_2(u, v) \tag{20}$$

By definition,

$$w_2(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} du \, dv \ W_2(u, v) \ e^{j(xu + yv)}$$

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Whence

$$w_2(x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du\ e^{jxu} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dv\ W_2(u,v) \right] \tag{21}$$

But from Eq. (16),

$$w_2(x,0) = w(x)$$

Therefore from Eq. (21), $\dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} dv\ W_2(u,v)$

is the Fourier transform of $w(x)$. This established Eq. (20).

Now,

$$W(u) \otimes H(u) = \int_{-\infty}^{\infty} dt\ W(t)\ H(u-t)$$

$$= \int_{-\infty}^{u} dt\ W(t) \tag{22}$$

and

$$W_2(u,v) \otimes H_2(u,v) = \int_{-\infty}^{\infty}\!\!\int dt ds\ W(t,s)\ H_2(u-t,v-s)$$

$$= \int_{-\infty}^{u} dt \int_{-\infty}^{\infty} ds\ W(t,s) \tag{23}$$

From Eqs. (22) and (20), we have

$$W(u) \otimes H(u) = \frac{1}{2\pi} \int_{-\infty}^{u} dt \int_{-\infty}^{\infty} ds\ W(t,s)$$

$$= \frac{1}{2\pi}\ W_2(u,v) \otimes H_2(u,v)$$

by virtue of Eq. (23).

IV.   The Design of Two-Dimensional Non-recursive Ideal Low-Pass Filters

One way of designing one-dimensional non-recursive digital filters is the so-called window method[7]. To fix ideas, let us consider the design of an ideal low-pass filter. The ideal frequency response is

$$F(u) = \begin{cases} 1, & \text{for } |u| \leq B \\ 0, & \text{for } |u| > B \end{cases} \tag{24}$$

where B is a constant. (This frequency response is actually repeated periodically because the impulse response is sampled). The inverse Fourier transform $f(x)$ of $F(u)$, which is the impulse response, has infinite duration, but in reality we have to use a finite-duration impulse response. So we truncate $f(x)$ by using a window:

$$g(x) = f(x) \ w(x) \tag{25}$$

$$w(x) = 0, \text{ for } |x| > A \tag{26}$$

where A is a constant. The actual frequency response we are getting is then

$$G(u) = F(u) \otimes W(u) \tag{27}$$

Suppose now we wish to design a two-dimensional ideal low-pass filter with an ideal Frequency response

$$F_2(u,v) = F(\sqrt{u^2 + v^2}) = \begin{cases} 1, & \text{for } |u^2 + v^2| \leq B^2 \\ 0, & \text{for } |u^2 + v^2| > B^2 \end{cases} \tag{28}$$

And we truncate the two-dimensional impulse response $f_2(x,y)$ by a two-dimensional window $w_2(x,y)$. Then the actual frequency response we are getting is

$$G_2(u,v) = F_2(u,v) \otimes W_2(u,v) \tag{29}$$

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Let us assume that the widths of $W(u)$ and $W_2(u,v)$ are much smaller than B, then near the discontinuities of $F(u)$, viz., $u = \pm B$, $G(u)$ is essentially equal to the convolution of $W(u)$ and a one-dimensional step function, and similarly, near the discontinuities of $F_2(u,v)$, viz., $u^2 + v^2 = B^2$, $G_2(u,v)$ is essentially equal to the convolution of $W_2(u,v)$ and a two-dimensional step function. It therefore follows from our theorem that:

if

$$w_2(x,y) = w \left(\sqrt{x^2 + y^2} \right) \tag{30}$$

then

$$\frac{1}{2\pi} G_2(u,v) \approx G \left(\sqrt{u^2 + v^2} \right) \tag{31}$$

This means that we can design a good two-dimensional low-pass filter by using the window $w_2(x,y)$ as given by Eq. (30), if $w(x)$ is a good window to use in designing a good one-dimensional low-pass filter.

V.   Summary

We have shown that if $w(x)$ is a good symmetrical one-dimensional window, then $w_2(x,y) = w \left(\sqrt{x^2 + y^2} \right)$ is a good circularly symmetrical two-dimensional window.

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